Developing the Equations of the Orbit

This chapter is about how earth orbit is achieved, the laws that describe the motion of an object orbiting another body, how satellites maneuver in space, and the determination of the look angle to a satellite from the earth using ephemeris data that describe the orbital trajectory of the satellite.

To achieve a stable orbit around the earth, a spacecraft must first be beyond the bulk of the earth's atmosphere, i.e., in what is popularly called space. There are many definitions of space. U.S. astronauts are awarded their "space wings" if they fly at an altitude that exceeds 50 miles (~80 km); some international treaties hold that the space frontier above a given country begins at a height of 100 miles (~160 km). Below 100 miles, permission must be sought to over-fly any portion of the country in question. On reentry, atmospheric drag starts to be felt at a height of about 400,000 ft (~76 miles ≈ 122 km). Most satellites, for any mission of more than a few months, are placed into orbits of at least 250 miles (≈400 km) above the earth. Even at this height, atmospheric drag is significant. As an example, the initial payload elements of the International Space Station (ISS) were injected into orbit at an altitude of 397 km when the shuttle mission left those modules on 9 June 1999. By the end of 1999, the orbital height had decayed to about 360 km, necessitating a maneuver to raise the orbit. Without onboard thrusters and sufficient orbital maneuvering fuel, the ISS would not last more than a few years at most in such a low orbit. To appreciate the basic laws that govern celestial mechanics, we will begin first with the fundamental Newtonian equations that describe the motion of a body. We will then give some coordinate axes within which the orbit of the satellite can be set and determine the various forces on the earth satellite.

Newton's laws of motion can be encapsulated into four equations:

$$s = ut + (\tfrac{1}{2})at^2 \tag{2.1a}$$

$$v^2 = u^2 + 2at \tag{2.1b}$$

$$v = u + at \tag{2.1c}$$

$$P = ma \tag{2.1d}$$

where s is the distance traveled from time $t = 0$; u is the initial velocity of the object at time $t = 0$ and v the final velocity of the object at time t; a is the acceleration of the object; P is the force acting on the object; and m is the mass of the object. Note that the acceleration can be positive or negative, depending on the direction it is acting with respect to the velocity vector. Of these four equations, it is the last one that helps us understand the motion of a satellite in a stable orbit (neglecting any drag or other perturbing forces). Put into words, Eq. (2.1d) states that the force acting on a body is equal to the mass of the body multiplied by the resulting acceleration of the body. Alternatively, the resulting acceleration is the ratio of the force acting on the body to the mass of the body. Thus, for a given force, the lighter the mass of the body, the higher the acceleration will be. When in a stable orbit, there are two main forces acting on a satellite: a centrifugal force due to the kinetic energy of the satellite, which attempts to fling the satellite into a higher orbit, and a centripetal force due to the gravitational attraction of the planet about which the satellite is orbiting, which attempts to pull the satellite down toward the planet. If these two forces are equal, the satellite will remain in a stable orbit. It will continually fall toward the planet's surface as it moves forward in its orbit but, by virtue of its orbital velocity, it will have moved forward just far enough to compensate for the "fall" toward the planet and so it will remain at the same orbital height. This is why an object in a stable orbit is sometimes described as being in "free fall." Figure 2.1 shows the two opposing forces on a satellite in a stable orbit[1].

Force = mass × acceleration and the unit of force is a Newton, with the notation *N*. A Newton is the force required to accelerate a mass of 1 kg with an acceleration of 1 m/s^2. The underlying units of a Newton are therefore (kg) × m/s^2. In Imperial Units one Newton = 0.2248 ft lb. The standard acceleration due to gravity at the earth's surface is 9.80665×10^{-3} km/s^2, which is often quoted as 981 cm/s^2. This value decreases

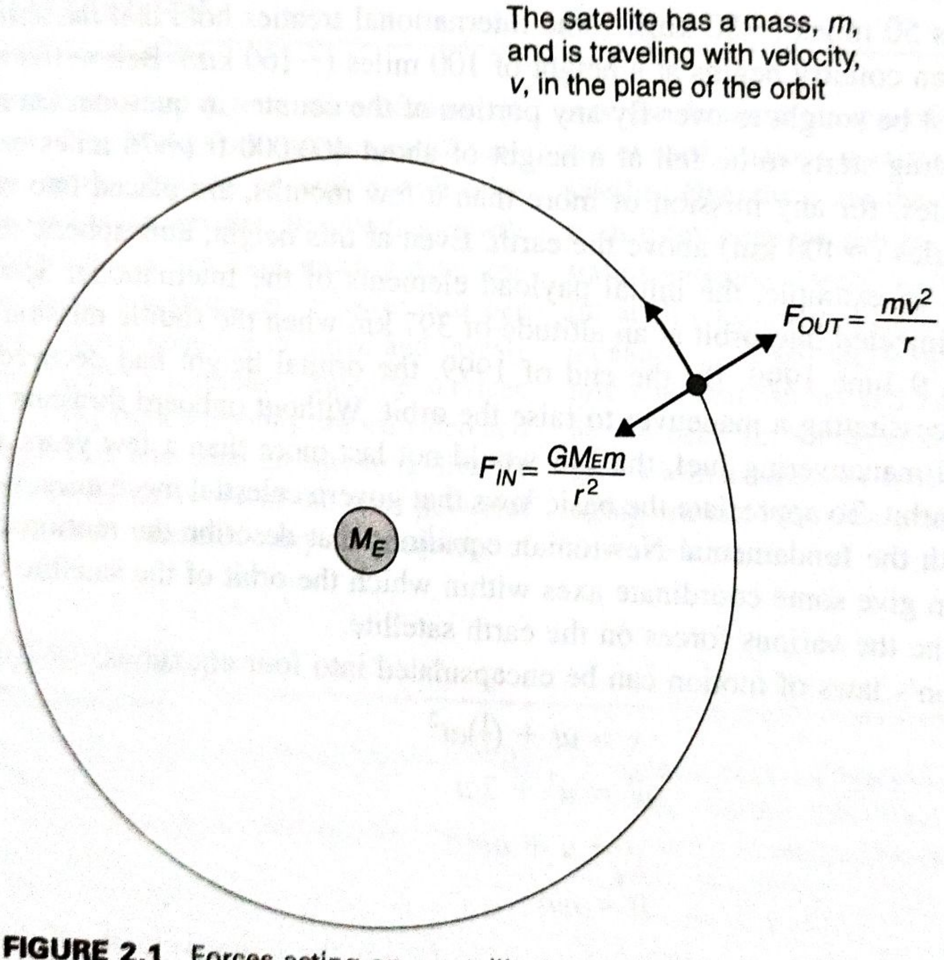

FIGURE 2.1 Forces acting on a satellite in a stable orbit around the earth (from Fig. 3.4 of reference 1). Gravitational force is inversely proportional to the square of the distance between the centers of gravity of the satellite and the planet the satellite is orbiting, in this case the earth. The gravitational force inward (F_{IN}, the centripetal force) is directed toward the center of gravity of the earth. The kinetic energy of the satellite (F_{OUT}, the centrifugal force) is directed diametrically opposite to the gravitational force. Kinetic energy is proportional to the square of the velocity of the satellite. When these inward and outward forces are balanced, the satellite moves around the earth in a "free fall" trajectory: the satellite's orbit. For a description of the units, please see the text.

with height above the earth's surface. The acceleration, a, due to gravity at a distance r from the center of the earth is[1]

$$a = \mu/r^2 \text{ km/s}^2 \tag{2.1}$$

where the constant μ is the product of the universal gravitational constant G and the mass of the earth M_E.

The product GM_E is called Kepler's constant and has the value 3.986004418×10^5 km^3/s^2. The universal gravitational constant is $G = 6.672 \times 10^{-11}$ Nm^2/kg^2 or 6.672×10^{-20} km^3/kg s^2 in the older units. Since force = mass × acceleration, the centripetal force acting on the satellite, F_{IN}, is given by

$$F_{IN} = m \times (\mu/r^2) \tag{2.2a}$$

$$= m \times (GM_E/r^2) \tag{2.2b}$$

In a similar fashion, the centrifugal acceleration is given by[1]

$$a = v^2/r \tag{2.3}$$

which will give the centrifugal force, F_{OUT}, as

$$F_{OUT} = m \times (v^2/r) \tag{2.4}$$

If the forces on the satellite are balanced, $F_{IN} = F_{OUT}$ and, using Eqs. (2.2a) and (2.4),

$$m \times \mu/r^2 = m \times v^2/r$$

hence the velocity v of a satellite in a circular orbit is given by

$$v = (\mu/r)^{1/2} \tag{2.5}$$

If the orbit is circular, the distance traveled by a satellite in one orbit around a planet is $2\pi r$, where r is the radius of the orbit from the satellite to the center of the planet. Since distance divided by velocity equals time to travel that distance, the period of the satellite's orbit, T, will be

$$T = (2\pi r)/v = (2\pi r)/[(\mu/r)^{1/2}]$$

Giving

$$T = (2\pi r^{3/2})/(\mu^{1/2}) \tag{2.6}$$

Table 2.1 gives the velocity, v, and orbital period, T, for four satellite systems that occupy typical LEO, MEO, and GEO orbits around the earth. In each case, the orbits are

TABLE 2.1 Orbital Velocity, Height, and Period of Four Satellite Systems

Satellite system	Orbital height (km)	Orbital velocity (km/s)	Orbital period (h min s)
Intelsat (GEO)	35,786.03	3.0747	23 56 4.1
New-ICO (MEO)	10,255	4.8954	5 55 48.4
Skybridge (LEO)	1,469	7.1272	1 55 17.8
Iridium (LEO)	780	7.4624	1 40 27.0

Mean earth radius is 6378.137 km and GEO radius from the center of the earth is 42,164.17 km.

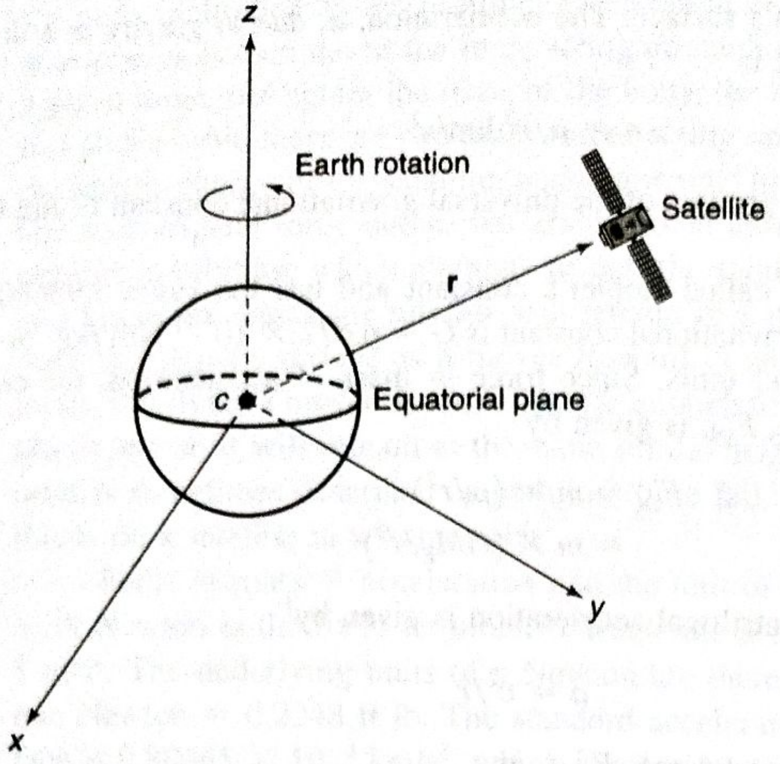

FIGURE 2.2 The initial coordinate system that could be used to describe the relationship between the earth and a satellite. A Cartesian coordinate system with the geographical axes of the earth as the principal axes is the simplest coordinate system to set up. The rotational axis of the earth is about the axis *cz*, where *c* is the center of the earth and *cz* passes through the geographic north pole. Axes *cx*, *cy* and *cz* are mutually orthogonal axes, with *cx* and *cy* passing through the earth's geographic equator. The vector ***r*** locates the moving satellite with respect to the center of the earth.

circular and the average radius of the earth is taken as 6378.137 km[1]. A number of coordinate systems and reference planes can be used to describe the orbit of a satellite around a planet. Figure 2.2 illustrates one of these using a Cartesian coordinate system with the earth at the center and the reference planes coinciding with the equator and the polar axis. This is referred to as a geocentric coordinate system.

With the coordinate system set up as in Figure 2.2, and with the satellite mass m located at a vector distance $\boldsymbol{r}$ from the center of the earth, the gravitational force $\boldsymbol{F}$ on the satellite is given by

$$\overline{F} = -\frac{GM_{\mathrm{E}}m\overline{r}}{r^3} \tag{2.7}$$

Where M_{E} is the mass of the earth and $G = 6.672 \times 10^{-11}$ Nm^2/kg^2. But force = mass × acceleration and Eq. (2.7) can be written as

$$\overline{F} = m\frac{d^2\overline{r}}{dt^2} \tag{2.8}$$

From Eqs. (2.7) and (2.8) we have

$$-\frac{\overline{r}}{r^3}\mu = \frac{d^2\overline{r}}{dt^2} \tag{2.9}$$

Which yields

$$\frac{d^2\overline{r}}{dt^2} + \frac{\overline{r}}{r^3}\mu = 0 \tag{2.10}$$

This is a second-order linear differential equation and its solution will involve six undetermined constants called the *orbital elements*. The orbit described by these orbital elements can be shown to lie in a plane and to have a constant angular momentum. The solution to Eq. (2.10) is difficult since the second derivative of $\boldsymbol{r}$ involves the second derivative of the unit vector $\boldsymbol{r}$. To remove this dependence, a different set of coordinates can

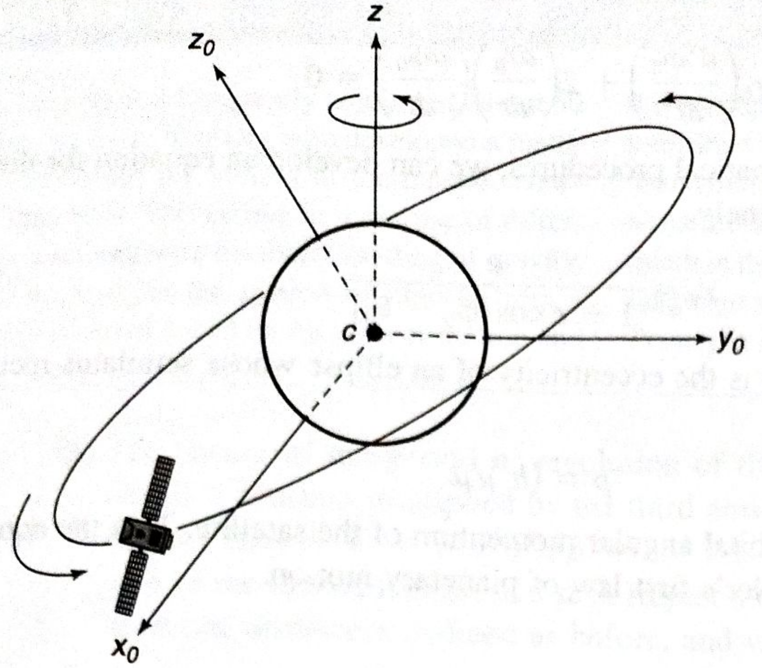

FIGURE 2.3 The orbital plane coordinate system. In this coordinate system, the orbital plane of the satellite is used as the reference plane. The orthogonal axes x_0 and y_0 lie in the orbital plane. The third axis, z_0, is perpendicular to the orbital plane. The geographical z-axis of the earth (which passes through the true North Pole and the center of the earth, c) does not lie in the same direction as the z_0 axis except for satellite orbits that are exactly in the plane of the geographical equator.

be chosen to describe the location of the satellite such that the unit vectors in the three axes are constant. This coordinate system uses the plane of the satellite's orbit as the reference plane. This is shown in Figure 2.3.

Expressing Eq. (2.10) in terms of the new coordinate axes x_0, y_0, and z_0 gives

$$\hat{x}_0\left(\frac{d^2x_0}{dt^2}\right) + \hat{y}_0\left(\frac{d^2y_0}{dt^2}\right) + \frac{\mu(x_0\hat{x}_0 + y_0\hat{y}_0)}{(x_0^2 + y_0^2)^{3/2}} = 0 \tag{2.11}$$

Equation (2.11) is easier to solve if it is expressed in a polar coordinate system rather than a Cartesian coordinate system. The polar coordinate system is shown in Figure 2.4.

With the polar coordinate system shown in Figure 2.4 and using the transformations

$$x_0 = r_0 \cos\phi_0 \tag{2.12a}$$

$$y_0 = r_0 \sin\phi_0 \tag{2.12b}$$

$$\hat{x}_0 = \hat{r}_0 \cos\phi_0 - \hat{\phi}_0 \sin\phi_0 \tag{2.12c}$$

$$\hat{y}_0 = \hat{\phi}_0 \cos\phi_0 + \hat{r}_0 \sin\phi_0 \tag{2.12d}$$

and equating the vector components of r_0 and ϕ_0 in turn in Eq. (2.11) yields

$$\frac{d^2r_0}{dt^2} - r_0\left(\frac{d\phi_0}{dt}\right) = -\frac{\mu}{r_0^2} \tag{2.13}$$

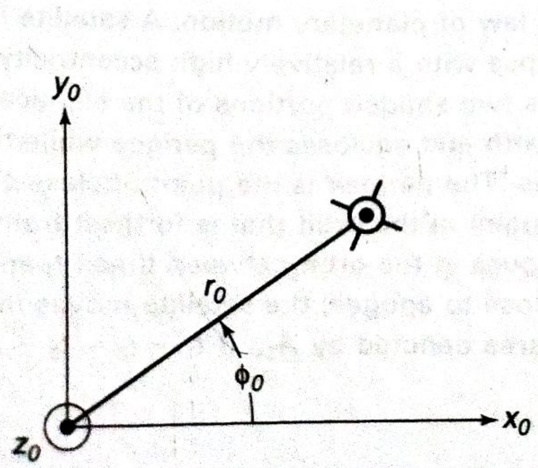

FIGURE 2.4 Polar coordinate system in the plane of the satellite's orbit. The plane of the orbit coincides with the plane of the paper. The axis z_0 is straight out of the paper from the center of the earth, and is normal to the plane of the satellite's orbit. The satellite's position is described in terms of the radius from the center of the earth r_0 and the angle this radius makes with the x_0 axis, ϕ_0.

and

$$r_0\left(\frac{d^2\phi_0}{dt^2}\right) + 2\left(\frac{dr_0}{dt}\right)\left(\frac{d\phi_0}{dt}\right) = 0 \tag{2.14}$$

Using standard mathematical procedures, we can develop an equation for the radius of the satellite's orbit, r_0, namely

$$r_0 = \frac{p}{1 + e\cos(\phi_0 - \theta_0)} \tag{2.15}$$

Where θ_0 is a constant and e is the eccentricity of an ellipse whose semilatus rectum p is given by

$$p = (h^2)/\mu \tag{2.16}$$

and h is magnitude of the orbital angular momentum of the satellite. That the equation of the orbit is an ellipse is Kepler's first law of planetary motion.

Kepler's Three Laws of Planetary Motion

Johannes Kepler (1571–1630) was a German astronomer and scientist who developed his three laws of planetary motion by careful observations of the behavior of the planets in the solar system over many years, with help from some detailed planetary observations by the Hungarian astronomer Tycho Brahe. Kepler's three laws are

1. The orbit of any smaller body about a larger body is always an ellipse, with the center of mass of the larger body as one of the two foci.
2. The orbit of the smaller body sweeps out equal areas in equal time (see Figure 2.5).

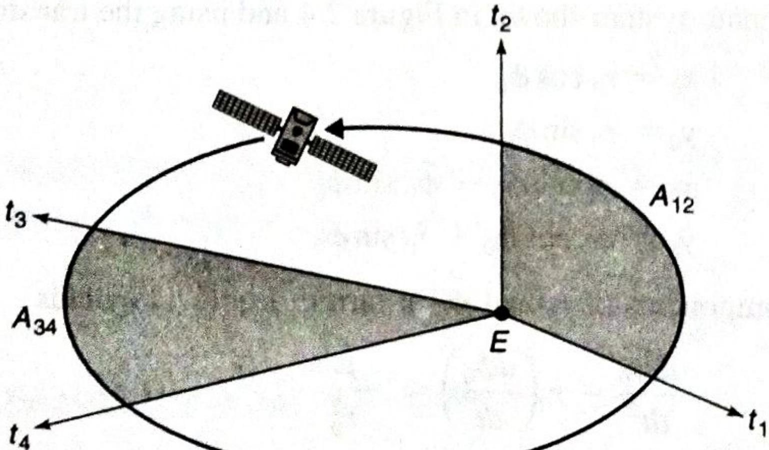

FIGURE 2.5 Illustration of Kepler's second law of planetary motion. A satellite is in orbit about the planet earth, E. The orbit is an ellipse with a relatively high eccentricity, that is, it is far from being circular. The figure shows two shaded portions of the elliptical plane in which the orbit moves, one is close to the earth and encloses the perigee while the other is far from the earth and encloses the apogee. The perigee is the point of closest approach to the earth while the apogee is the point in the orbit that is furthest from the earth. While close to perigee, the satellite moves in the orbit between times t_1 and t_2 and sweeps out an area denoted by A_{12}. While close to apogee, the satellite moves in the orbit between times t_3 and t_4 and sweeps out an area denoted by A_{34}. If $t_1 - t_2 = t_3 - t_4$ then $A_{12} = A_{34}$.

SIDEBAR

Kepler's laws were subsequently confirmed, about 50 years later, by Isaac Newton, who developed a mathematical model for the motion of the planets. Newton was one of the first people to make use of differential calculus, and with his understanding of gravity, was able to describe the motion of planets from a mathematical model based on his laws of motion and the concept of gravitational attraction. The work was published in the *Philosophiae Naturalis Principia Mathematica* in 1687. At that time, Latin was the international language of formally educated people, much in the way English has become the international language of e-mail and business today, so Newton's *Principia* was written in Latin.

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3. The square of the period of revolution of the smaller body about the larger body equals a constant multiplied by the third power of the semimajor axis of the orbital ellipse. That is, $T^2 = (4\pi^2 a^3)/\mu$ where T is the orbital period, a is the semimajor axis of the orbital ellipse, and μ is Kepler's constant. If the orbit is circular, then a becomes distance r, defined as before, and we have Eq. (2.6).

Describing the orbit of a satellite enables us to develop Kepler's second two laws.

Describing the Orbit of a Satellite

The quantity θ_0 in Eq. (2.15) serves to orient the ellipse with respect to the orbital plane axes x_0 and y_0. Now that we know that the orbit is an ellipse, we can always choose x_0 and y_0 so that θ_0 is zero. We will assume that this has been done for the rest of this discussion. This now gives the equation of the orbit as

$$r_0 = \frac{p}{1 + e\cos\phi_0} \tag{2.17}$$

The path of the satellite in the orbital plane is shown in Figure 2.6. The lengths a and b of the semimajor and semiminor axes are given by

$$a = p/(1 - e^2) \tag{2.18}$$

$$b = a(1 - e^2)^{1/2} \tag{2.19}$$

The point in the orbit where the satellite is closest to the earth is called the *perigee* and the point where the satellite is farthest from the earth is called the *apogee*. The perigee and apogee are **always** exactly oppo··te each other. To make θ_0 equal to zero, we have chosen the x_0 axis so that both the apogee and the perigee lie along it and the x_0 axis is therefore the major axis of the ellipse.

The differential area swept out by the vector $\mathbf{r_0}$ from the origin to the satellite in time dt is given by

$$dA = 0.5r_0^2\left(\frac{d\phi_0}{dt}\right)dt = 0.5h\,dt \tag{2.20}$$

Remembering that h is the magnitude of the orbital angular momentum of the satellite, the radius vector of the satellite can be seen to sweep out equal areas in equal times. This is Kepler's second law of planetary motion. By equating the area of the ellipse (πab) to the area swept out in one orbital revolution, we can derive an expression for the orbital period T as

$$T^2 = (4\pi^2 a^3)/\mu \tag{2.21}$$

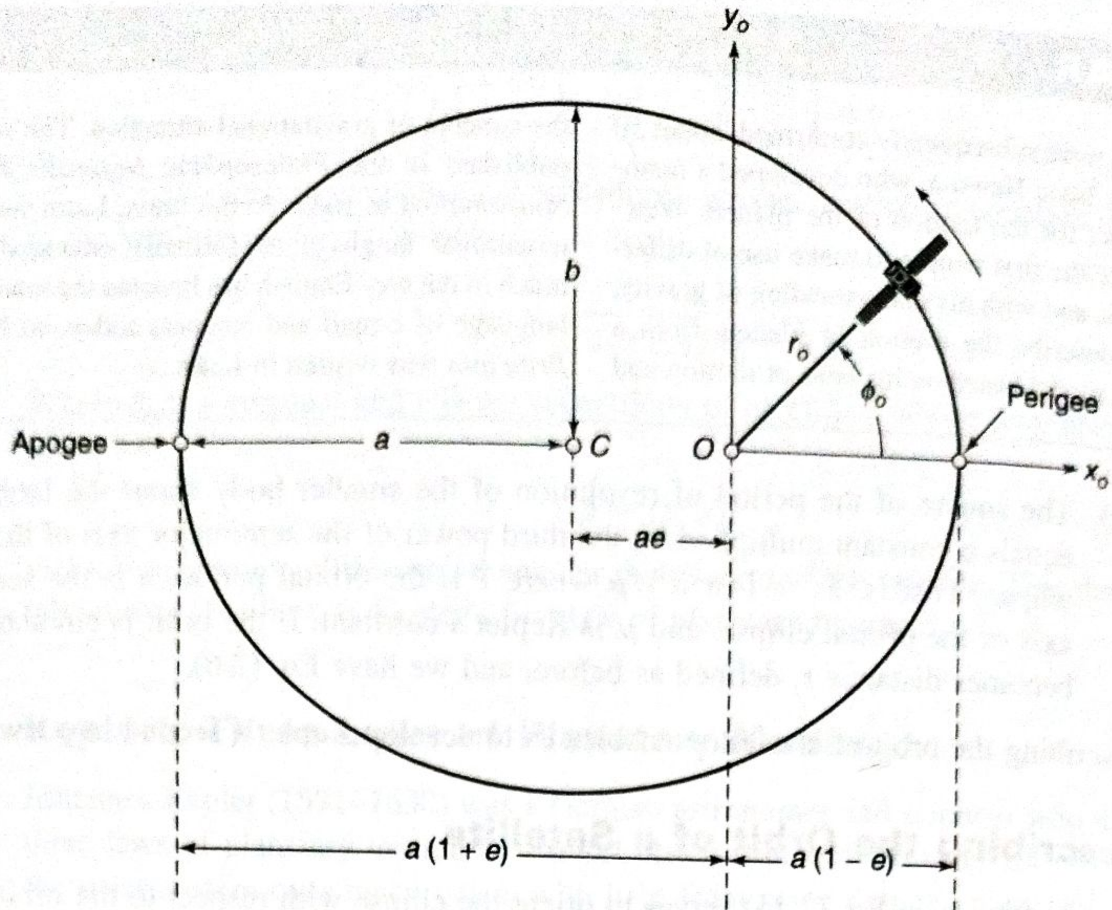

FIGURE 2.6 The orbit as it appears in the orbital plane. The point *O* is the center of the earth and the point *C* is the center of the ellipse. The two centers do not coincide unless the eccentricity, *e*, of the ellipse is zero (i.e., the ellipse becomes a circle and $a = b$). The dimensions *a* and *b* are the semimajor and semiminor axes of the orbital ellipse, respectively.

This equation is the mathematical expression of Kepler's third law of planetary motion: the square of the period of revolution is proportional to the cube of the semimajor axis. (Note that this is the square of Eq. (2.6) and that in Eq. (2.6) the orbit was assumed to be circular such that semimajor axis a = semiminor axis b = circular orbit radius from the center of the earth r.) *Kepler's third law extends the result from Eq. (2.6), which was derived for a circular orbit, to the more general case of an elliptical orbit.* Equation (2.21) is extremely important in satellite communications systems. This equation determines the period of the orbit of any satellite, and it is used in every GPS receiver in the calculation of the positions of GPS satellites. Equation (2.21) is also used to find the orbital radius of a GEO satellite, for which the period T must be made exactly equal to the period of one revolution of the earth for the satellite to remain stationary over a point on the equator.

An important point to remember is that the period of revolution, T, is referenced to inertial space, namely, to the galactic background. The orbital period is the time the orbiting body takes to return to the same reference point in space with respect to the galactic background. Nearly always, the primary body will also be rotating and so the period of revolution of the satellite may be different from that perceived by an observer who is standing still on the surface of the primary body. This is most obvious with a geostationary earth orbit (GEO) satellite (see Table 2.1). The orbital period of a GEO satellite is exactly equal to the period of rotation of the earth, 23 h 56 min 4.1 s, but, to an observer on the ground, the satellite appears to have an infinite orbital period: it always stays in the same place in the sky.

To be perfectly geostationary, the orbit of a satellite needs to have three features: (a) it must be exactly circular (i.e., have an eccentricity of zero); (b) it must be at the correct altitude (i.e., have the correct period); and (c) it must be in the plane of the equator (i.e., have a zero inclination with respect to the equator). If the inclination of the satellite is not zero and/or if the eccentricity is not zero, but the orbital period is correct, then the satellite will be in a *geosynchronous* orbit. The position of a geosynchronous satellite will appear to oscillate about a mean look angle in the sky with respect to a stationary observer on the earth's surface. The orbital period of a GEO satellite, 23 h 56 min 4.1 s, is one sidereal day. A sidereal day is the time between consecutive crossings of any particular longitude on the earth by any star, other than the sun[1]. The mean solar day of 24 h is the time between any consecutive crossings of any particular longitude by the sun, and is the time between successive sunrises (or sunsets) observed at one location on earth, averaged over an entire year. Because the earth moves round the sun once per 365 ¼ days, the solar day is 1440/365.25 = 3.94 min longer than a sidereal day.

Locating the Satellite in the Orbit

Consider now the problem of locating the satellite in its orbit. The equation of the orbit may be rewritten by combining Eqs. (2.15) and (2.18) to obtain

$$r_0 = \frac{a(1 - e^2)}{1 + e \cos \phi_0} \tag{2.22}$$

The angle ϕ_0 (see Figure 2.6) is measured from the x_0 axis and is called the *true anomaly*. [*Anomaly* was a measure used by astronomers to mean a planet's angular distance from its perihelion (closest approach to the sun), measured as if viewed from the sun. The term was adopted in celestial mechanics for all orbiting bodies.] Since we defined the positive x_0 axis so that it passes through the perigee, ϕ_0 measures the angle from the perigee to the instantaneous position of the satellite. The rectangular coordinates of the satellite are given by

$$x_0 = r_0 \cos \phi_0 \tag{2.23}$$

$$y_0 = r_0 \sin \phi_0 \tag{2.24}$$

As noted earlier, the orbital period T is the time for the satellite to complete a revolution in inertial space, traveling a total of 2π radians. The average angular velocity η is thus

$$\eta = (2\pi)/T = (\mu^{1/2})/(a^{3/2}) \tag{2.25}$$

If the orbit is an ellipse, the instantaneous angular velocity will vary with the position of the satellite around the orbit. If we enclose the elliptical orbit with a *circumscribed circle* of radius a (see Figure 2.7), then an object going around the circumscribed circle with a constant angular velocity η would complete one revolution in exactly the same period T as the satellite requires to complete one (elliptical) orbital revolution.

Consider the geometry of the circumscribed circle as shown in Figure 2.7. Locate the point (indicated as A) where a vertical line drawn through the position of the satellite intersects the circumscribed circle. A line from the center of the ellipse (C) to this point (A) makes an angle E with the x_0 axis; E is called the *eccentric anomaly* of the satellite.

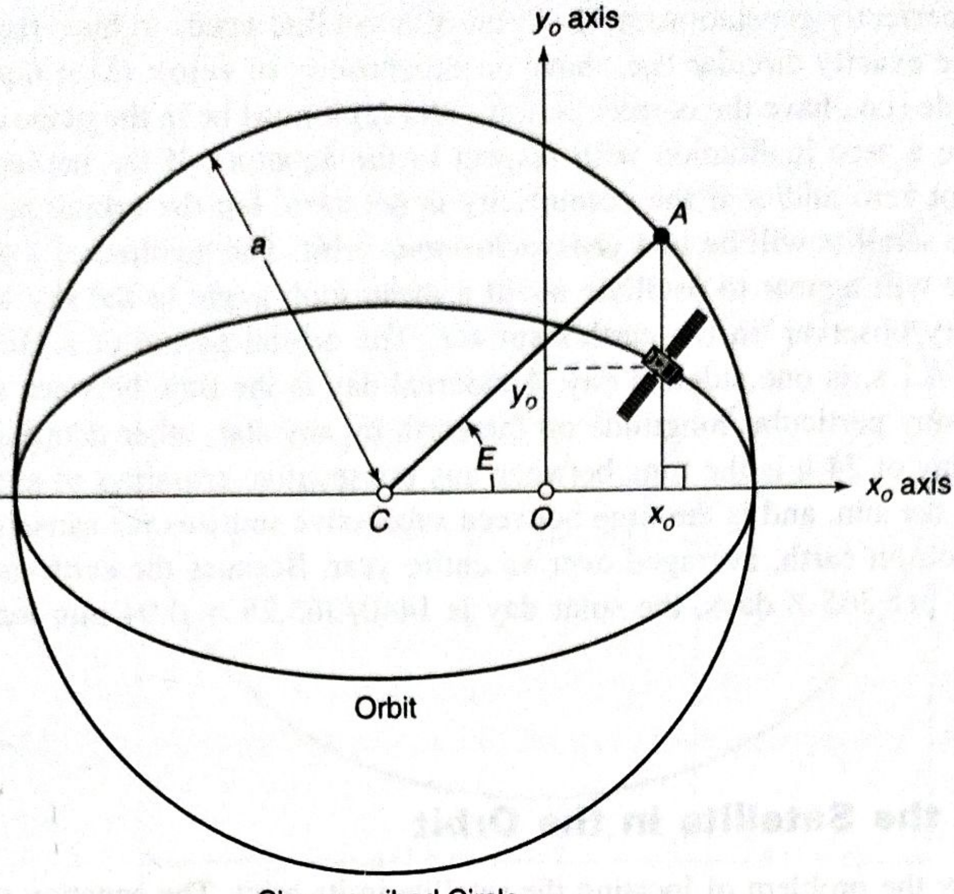

FIGURE 2.7 The circumscribed circle and the eccentric anomaly E. Point O is the center of the earth and point C is both the center of the orbital ellipse and the center of the circumscribed circle. The satellite location in the orbital plane coordinate system is specified by (x_0, y_0). A vertical line through the satellite intersects the circumscribed circle at point A. The eccentric anomaly E is the angle from the x_0 axis to the line joining C and A.

It is related to the radius r_0 by

$$r_0 = a(1 - e\cos E) \tag{2.26}$$

Thus

$$a - r_0 = ae\cos E \tag{2.27}$$

We can also develop an expression that relates eccentric anomaly E to the average angular velocity η, which yields

$$\eta\, dt = (1 - e\cos E)dE \tag{2.28}$$

Let t_p be the *time of perigee*. This is simultaneously the time of closest approach to the earth; the time when the satellite is crossing the x_0 axis; and the time when E is zero. If we integrate both sides of Eq. (2.28), we obtain

$$\eta(t - t_p) = E - e\sin E \tag{2.29}$$

The left side of Eq. (2.29) is called the *mean anomaly*, M. Thus

$$M = \eta(t - t_p) = E - e\sin E \tag{2.30}$$

The mean anomaly M is the arc length (in radians) that the satellite would have traversed since the perigee passage if it were moving on the circumscribed circle at the mean angular velocity η.

If we know the time of perigee, t_p, the eccentricity, e, and the length of the semimajor axis, a, we now have the necessary equations to determine the coordinates (r_0, ϕ_0)

and (x_0, y_0) of the satellite in the orbital plane. The process is as follows

1. Calculate η using Eq. (2.25).
2. Calculate M using Eq. (2.30).
3. Solve Eq. (2.30) for E.
4. Find r_0 from E using Eq. (2.27).
5. Solve Eq. (2.22) for ϕ_0.
6. Use Eqs. (2.23) and (2.24) to calculate x_0 and y_0.

Now we must locate the orbital plane with respect to the earth.

Locating the Satellite with Respect to the Earth

At the end of the last section, we summarized the process for locating the satellite at the point (x_0, y_0, z_0) in the rectangular coordinate system of the orbital plane. The location was with respect to the center of the earth. In most cases, we need to know where the satellite is from an observation point that is not at the center of the earth. We will therefore develop the transformations that permit the satellite to be located from a point on the rotating surface of the earth. We will begin with a *geocentric equatorial coordinate system* as shown in Figure 2.8. The rotational axis of the earth is the z_i axis, which is through the geographic North Pole. The x_i axis is from the center of the earth toward a fixed location in space called the *first point of Aries* (see Figure 2.8). This coordinate system moves through space; it translates as the earth moves in its orbit around the sun, but it does not rotate as the earth rotates. The x_i direction is always the same, whatever the earth's position around the sun and is in the direction of the first point of Aries. The (x_i, y_i) plane contains the earth's equator and is called the *equatorial plane*.

Angular distance measured eastward in the equatorial plane from the x_i axis is called *right ascension* and given the symbol *RA*. The two points at which the orbit

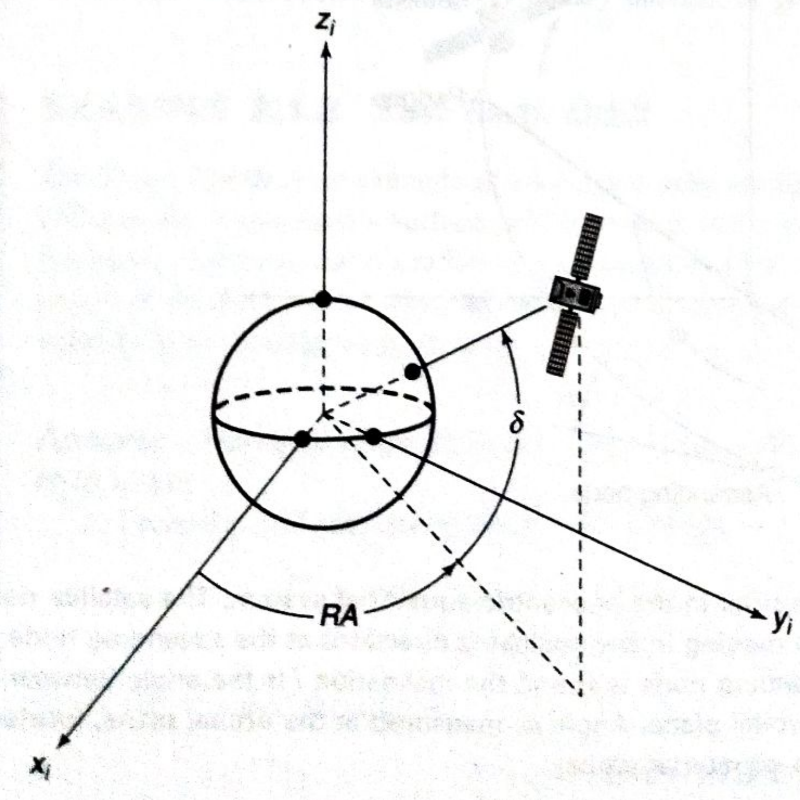

FIGURE 2.8 The geocentric equatorial system. This geocentric system differs from that shown in Figure 2.1 only in that the x_i axis points to the first point of Aries. The first point of Aries is the direction of a line from the center of the earth through the center of the sun at the vernal equinox (about March 21 in the Northern Hemisphere), the instant when the subsolar point crosses the equator from south to north. In the above system, an object may be located by its right ascension *RA* and its declination δ.

penetrates the equatorial plane are called nodes; the satellite moves upward throug the equatorial plane at the *ascending node* and downward through the equatoria plane at the *descending node*, given the conventional picture of the earth, with nor at the top, which is in the direction of the positive z axis for the earth centered coo dinate set. Remember that in space there is no *up* or *down*; that is a concept we ar familiar with because of gravity at the earth's surface. For a weightless body in spac such as an orbiting spacecraft, up and down have no meaning unless they are define with respect to a reference point. The *right ascension of the ascending node* is calle Ω. The angle that the orbital plane makes with the equatorial plane (the planes inte sect at the line joining the nodes) is called the *inclination*, i. Figure 2.9 illustrates the quantities.

The variables Ω and i together locate the orbital plane with respect to the equato rial plane. To locate the orbital coordinate system with respect to the equatorial coordi nate system we need ω, the *argument of perigee west*. This is the angle measured along the orbit from the ascending node to the perigee.

Standard time for space operations and most other scientific and engineering pur poses is *universal time* (UT), also known as *zulu time* (z). This is essentially the mea solar time at the Greenwich Observatory near London, England. Universal time is meas ured in hours, minutes, and seconds or in fractions of a day. It is 5 h later than Easter Standard Time, so that 07:00 EST is 12:00:00 h UT. The civil or calendar day begin at 00:00:00 hours UT, frequently written as 0 h. This is, of course, midnight (24:00:00 on the previous day. Astronomers employ a second dating system involving *Julian day* and *Julian dates*. Julian days start at noon UT in a counting system whereby noon on December 31, 1899, was the beginning of Julian day 2415020, usually written 241 5020. These are extensively tabulated in reference 2 and additional information is in reference 14. As an example, noon on December 31, 2000, the eve of the twenty-first century, is the start of Julian day 245 1909. Julian dates can be used to indicate time by append- ing a decimal fraction; 00:00:00 h UT on January 1, 2001—zero hour, minute, and

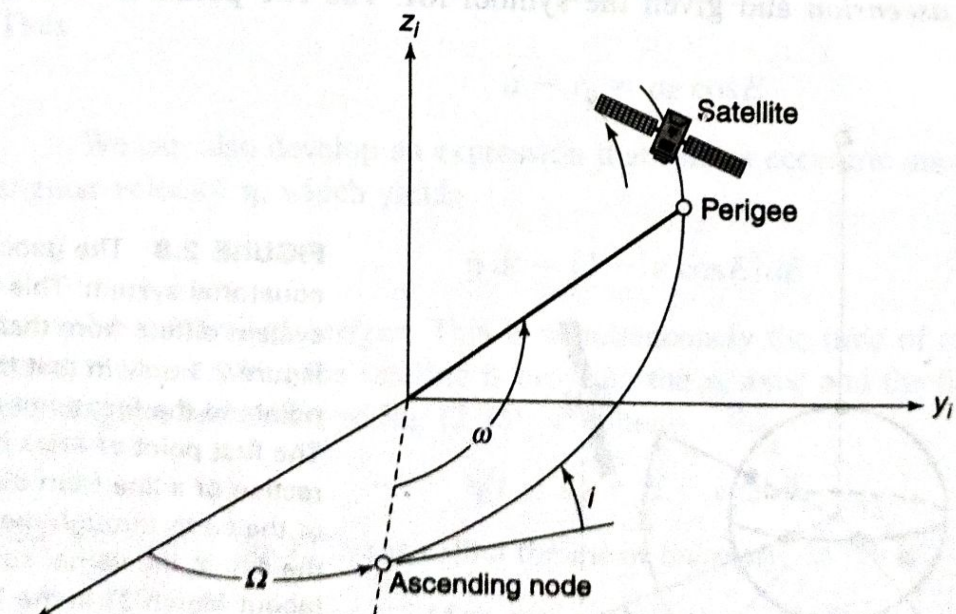

FIGURE 2.9 Locating the orbit in the geocentric equatorial system. The satellite penetrates the equatorial plane (while moving in the positive z direction) at the ascending node. The right ascension of the ascending node is Ω and the inclination i is the angle between the equatorial plane and the orbital plane. Angle ω, measured in the orbital plane, locates the perigee with respect to the equatorial plane.

second for the third millenium A.D.—is given by Julian date 245 1909.5. To find the exact position of an orbiting satellite at a given instant in time requires knowledge of the orbital elements.

Orbital Elements

To specify the absolute (i.e., the inertial) coordinates of a satellite at time t, we need to know six quantities. (This was evident earlier when we determined that a satellite's equation of motion was a second order vector linear differential equation.) These quantities are called the orbital elements. More than six quantities can be used to describe a unique orbital path and there is some arbitrariness in exactly which six quantities are used. We have chosen to adopt a set that is commonly used in satellite communications: eccentricity (e), semimajor axis (a), time of perigee (t_p), right ascension of ascending node (Ω), inclination (i), and argument of perigee (ω). Frequently, the mean anomaly (M) at a given time is substituted for t_p.

EXAMPLE 2.1.1 *Geostationary Satellite Orbit Radius*

The earth rotates once per sidereal day of 23 h 56 min 4.09 s. Use Eq. (2.21) to show that the radius of the GEO is 42,164.17 km as given in Table 2.1.

Answer Equation (2.21) gives the square of the orbital period in seconds

$$T^2 = (4\pi^2 a^3)/\mu$$

Rearranging the equation, the orbital radius a is given by

$$a^3 = T^2\mu/(4\pi^2)$$

For one sidereal day, $T = 86{,}164.09$ s. Hence

$$a^3 = (86{,}164.1)^2 \times 3.986004418 \times 10^5/(4\pi^2) = 7.496020251 \times 10^{13} \text{ km}^3$$

$$a = 42{,}164.17 \text{ km}$$

This is the orbital radius for a geostationary satellite, as given in Table 2.1. ■

EXAMPLE 2.1.2 *Low Earth Orbit*

The Space Shuttle is an example of a low earth orbit satellite. Sometimes, it orbits at an altitude of 250 km above the earth's surface, where there is still a finite number of molecules from the atmosphere. The mean earth's radius is approximately 6378.14 km. Using these figures, calculate the period of the shuttle orbit when the altitude is 250 km and the orbit is circular. Find also the linear velocity of the shuttle along its orbit.

Answer The radius of the 250-km altitude Space Shuttle orbit is $(r_e + h) = 6378.14 + 250.0 =$ 6628.14 km

From Eq. 2.21, the period of the orbit is T where

$$T^2 = (4\pi^2 a^3)/\mu = 4\pi^2 \times (6628.14)^3/3.986004418 \times 10^5 \text{ s}^2$$

$$= 2.88401145 \times 10^7 \text{ s}^2$$

Hence the period of the orbit is

$$T = 5370.30 \text{ s} = 89 \text{ min } 30.3 \text{ s.}$$

This orbit period is about as small as possible. At a lower altitude, friction with the earth's atmosphere will quickly slow the Shuttle down and it will return to earth. Thus, all spacecraft in stable earth orbit have orbital periods exceeding 89 min 30 s.

The circumference of the orbit is $2\pi a = 41{,}645.83$ km.

Hence the velocity of the Shuttle in orbit is

$$2\pi a/T = 41{,}645.83/5370.13 = 7.755 \text{ km/s}$$

Alternatively, you could use Eq. (2.5): $v = (\mu/r)^{1/2}$. The term $\mu = 3.986004418 \times 10^5$ km^3/s^2 and the term $r = (6378.14 + 250.0)$ km, yielding $v = 7.755$ km/s.

Note: If μ and r had been quoted in units of m^3/s^2 and m, respectively, the answer would have been in meters/second. Be sure to keep the units the same during a calculation procedure.

A velocity of about 7.8 km/s is a typical velocity for a low earth orbit satellite. As the altitude of a satellite increases, its velocity becomes smaller. ■

EXAMPLE 2.1.3 *Elliptical orbit*

A satellite is in an elliptical orbit with a perigee of 1000 km and an apogee of 4000 km. Using a mean earth radius of 6378.14 km, find the period of the orbit in hours, minutes, and seconds, and the eccentricity of the orbit.

Answer The major axis of the elliptical orbit is a straight line between the apogee and perigee, as seen in Figure 2.7. Hence, for a semimajor axis length a, earth radius r_e, perigee height h_p, and apogee height h_a,

$$2a = 2r_e + h_p + h_a = 2 \times 6378.14 + 1000.0 + 4000.0 = 17{,}756.28 \text{ km}$$

Thus the semimajor axis of the orbit has a length $a = 8878.14$ km. Using this value of a in Eq. (2.21) gives an orbital period T seconds where

$$T^2 = (4\pi^2 a^3)/\mu = 4\pi^2 \times (8878.07)^3/3.986004418 \times 10^5 \text{ s}^2$$

$$= 6.930872802 \times 10^7 \text{ s}^2$$

$$T = 8325.1864 \text{ s} = 138 \text{ min } 45.19 \text{ s} = 2 \text{ h } 18 \text{ min } 45.19 \text{ s}$$

The eccentricity of the orbit is given by e, which can be found from Eq. (2.27) by considering the instant at which the satellite is at perigee. Referring to Figure 2.7, when the satellite is at perigee, the eccentric anomaly $E = 0$ and $r_0 = r_e + h_p$. From Eq. (2.27), at perigee

$$r_0 = a(1 - e\cos E) \quad \text{and} \quad \cos E = 1$$

Hence

$$r_e + h_p = a(1 - e)$$

$$e = 1 - (r_e + h_p)/a = 1 - 7{,}378.14/8878.14 = 0.169$$ ■