5.4. DIVERGENCE OF MAGNETIC INDUCTION, B

A current element $I\,dl$ at a source point $P'(x', y', z')$ produces an element of magnetic induction $d\mathbf{B}$ at a field point $P(x, y, z)$. According to Biot and Savart law,

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\,dl \times \mathbf{r}}{r^3}$$

We write, according to Biot and Savart law, that for entire current loop

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{dl \times \mathbf{r}}{r^3}$$

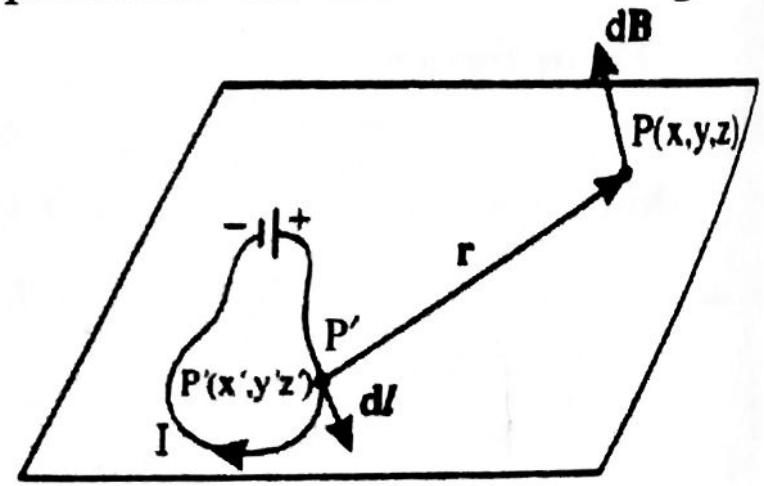

Fig. 13. Source point, P′ and field point.

and
$$\vec{\nabla}.\mathbf{B} = \frac{\mu_0 I}{4\pi}\vec{\nabla}.\oint \frac{dl\times\mathbf{r}}{r^3}, \qquad ...(1)$$

where **B** is the magnetic induction at the field $P(x, y, z)$ and the element of conductor dl carrying current I is at a source point $P'(x', y', z')$.

The derivatives in the divergence operator are calculated at the field point, the differentiation and integration operations are interchangeable, we write eq. (1) as

$$\vec{\nabla}.\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint \vec{\nabla}.\left(\frac{dl\times\mathbf{r}}{r^3}\right)$$

Further

$$\vec{\nabla}.(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.(\vec{\nabla}\times\mathbf{A}) - \mathbf{A}.(\vec{\nabla}\times\mathbf{B})$$

$$\vec{\nabla}.\mathbf{B} = \frac{\mu_0 I}{4\pi}\left\{\oint\left(\frac{\mathbf{r}}{r^3}\right).(\vec{\nabla}\times dl) - dl.\left(\vec{\nabla}\times\frac{\mathbf{r}}{r^3}\right)\right\} \qquad ...(2)$$

But we find that

$$\vec{\nabla}\times dl = 0,$$

because dl is not a function of coordinates (x, y, z) of field point P where we are to find $\vec{\nabla}.\mathbf{B}$. Also,

$$\vec{\nabla}\times\frac{\mathbf{r}}{r^3} = -\vec{\nabla}\times\vec{\nabla}\left(\frac{1}{r}\right)^{*} = 0,$$

because the curl of a gradient is always zero. Therefore eq. (2) becomes

$$\vec{\nabla}.\mathbf{B} = 0 \qquad ...(3)$$

that is, divergence of the magnetic induction **B** *is always zero.* This implies that

$$\phi = \int_S \mathbf{B}.d\mathbf{S}$$
$$= \int_{V'} (\vec{\nabla}.\mathbf{B})\,dV'$$
$$= 0,$$

the net flux of magnetic induction through any closed surface is always zero.

5.5 THE MAGNETIC VECTOR POTENTIAL, A

We know that electrostatic field intensity **E** can be drived from the potential V by the relation $\mathbf{E} = -\vec{\nabla}V$. Likewise we shall show that magnetic induction **B** can also be related to a quantity **A** by the relation $\mathbf{B} = \vec{\nabla}\times\mathbf{A}$, where A, by analogy, is called magnetic vector potential.

$$* \ \vec{\nabla}\left(\frac{1}{r}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$
$$= -\frac{\mathbf{i}}{r^2}\frac{\partial r}{\partial x} - \frac{\mathbf{j}}{r^2}\frac{\partial r}{\partial y} - \frac{\mathbf{k}}{r^2}\frac{\partial r}{\partial z}$$
$$= -\frac{\mathbf{i}}{r^2}\frac{\partial}{\partial x}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$
$$-\frac{\mathbf{j}}{r^2}\frac{\partial}{\partial y}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$
$$-\frac{\mathbf{k}}{r^2}\frac{\partial}{\partial z}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$
$$= -\frac{\mathbf{i}(x-x')}{r^3} - \frac{\mathbf{j}(y-y')}{r^3} - \frac{\mathbf{k}(z-z')}{r^3} = -\frac{\mathbf{r}}{r^3}$$

Refer to fig. 13. We write the magnetic induction as

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{l} \times \mathbf{r}}{r^3}$$

Also
$$\frac{r}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right)$$

where

$$\vec{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

involving coordinates of field point $P(x, y, z)$. Then we write

$$\mathbf{B} = -\frac{\mu_0 I}{4\pi} \oint d\boldsymbol{l} \times \vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \frac{\mu_0 I}{4\pi} \int \vec{\nabla}\left(\frac{1}{r}\right) \times d\boldsymbol{l}. \qquad \ldots(1)$$

We know that

$$\vec{\nabla} \times (p\mathbf{Q}) = p\,(\vec{\nabla} \times \mathbf{Q}) - (\mathbf{Q} \times \vec{\nabla} p)$$

or
$$(\vec{\nabla} p \times \mathbf{Q}) = \vec{\nabla} \times (p\mathbf{Q}) - p\,(\vec{\nabla} \times \mathbf{Q})$$

Putting $d\boldsymbol{l}$ for $\mathbf{Q}$ and $\left(\frac{1}{r}\right)$ for p, we write

$$\vec{\nabla}\left(\frac{1}{r}\right) \times d\boldsymbol{l} = \vec{\nabla} \times \left(\frac{d\boldsymbol{l}}{r}\right) - \frac{1}{r}(\vec{\nabla} \times d\boldsymbol{l}),$$

so that equation (1) is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\left[\oint \left(\vec{\nabla} \times \frac{d\boldsymbol{l}}{r}\right) - \oint \frac{1}{r}(\vec{\nabla} \times d\boldsymbol{l})\right]$$

But
$$\vec{\nabla} \times d\boldsymbol{l} = 0,$$

because $d\boldsymbol{l}$ is not the function of field point coordinates (x, y, z). Thus

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \left(\vec{\nabla} \times \frac{d\boldsymbol{l}}{r}\right) \qquad \ldots(2)$$

Interchanging the operations, we get

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \vec{\nabla} \times \left(\oint \frac{d\boldsymbol{l}}{r}\right)$$

$$= \vec{\nabla} \times \left(\frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{l}}{r}\right)$$

$$= \vec{\nabla} \times \mathbf{A}, \qquad \ldots(3)$$

where
$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{l}}{r},$$

called the vector potential. *Thus magnetic induction is given by the curl of vector potential.* If the current is distributed with a current density $\mathbf{J}$ so that

$$I = \mathbf{J}dS,$$

we get on putting $dS\,d\boldsymbol{l} = dV'$ and integrating over the whole volume,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}\,dV'}{r}, \qquad \ldots(4)$$

5.6. THE DIVERGENCE OF MAGNETIC VECTOR POTENTIAL, A : THE LORENTZ CONDITION

We shall calculate

$$\vec{\nabla}.\mathbf{A} = \frac{\mu_0}{4\pi}\vec{\nabla}.\int_{V'} \frac{\mathbf{J}\,dV'}{r} \qquad \ldots(1)$$

where **A** is evaluated at the field point $P(x, y, z)$ and volume element dV' is situated at source point $P'(x', y', z',)$ fig. 14. Since $\vec{\nabla}$ involves derivative of field point (x, y, z), we can change the operations in eq. (1). That is

$$\vec{\nabla}.\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \vec{\nabla}.\frac{\mathbf{J}\,dV'}{r}$$

$$= \frac{\mu_0}{4\pi}\left[\int_{V'} \frac{1}{r}(\vec{\nabla}.\,\mathbf{J})\,dV' + \int_{V'} \mathbf{J}.\vec{\nabla}\left(\frac{1}{r}\right)dV'\right]$$

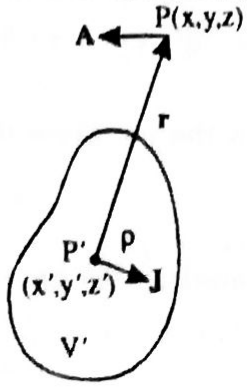

Fig. 14. A is to be evaluated at field point *P*, J is current density at source point P′ and ρ is charge density, *dV′* is the volume element at source point *P′*

Since **J** is current density at the source point P', therefore it will not be a function of x, y, z giving then

$$\vec{\nabla}.\mathbf{J} = 0$$

$$\vec{\nabla}.\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{J}.\vec{\nabla}\left(\frac{1}{r}\right)dV'. \qquad \ldots(2)$$

Setting the gradient at the source point to be $\vec{\nabla}'\left(\frac{1}{r}\right)$, we can show that

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\vec{\nabla}'\left(\frac{1}{r}\right)^{*}$$

Therefore
$$\vec{\nabla}.\mathbf{A} = -\frac{\mu_0}{4\pi}\int_{V'} \mathbf{J}.\vec{\nabla}'\left(\frac{1}{r}\right)dV'$$

$$= -\frac{\mu_0}{4\pi}\left(\int_{V'} \vec{\nabla}'.\frac{\mathbf{J}}{r}dV' - \frac{1}{r}(\vec{\nabla}'.\,\mathbf{J})\,dV'\right),$$

Changing with the help of divergence theorem, we put

$$\int_{V'} \vec{\nabla}.\frac{\mathbf{J}}{r}dV' = \int_S \frac{\mathbf{J}}{r}.d\mathbf{S}$$

$$* \ \vec{\nabla}'\left(\frac{1}{r}\right) = \mathbf{i}\frac{\partial}{\partial x'}\left(\frac{1}{r}\right) + \mathbf{j}\frac{\partial}{\partial y'}\left(\frac{1}{r}\right) + \mathbf{k}\frac{\partial}{\partial z'}\left(\frac{1}{r}\right)$$

$$= -\frac{\mathbf{i}}{r^2}\frac{\partial}{\partial x'}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$

$$-\frac{\mathbf{j}}{r^2}\frac{\partial}{\partial y'}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$

$$-\frac{\mathbf{k}}{r^2}\frac{\partial}{\partial z'}\left(\sqrt{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}}\right)$$

$$= +\frac{\mathbf{i}}{r^3}(x-x') + \frac{\mathbf{j}}{r^3}(y-y') + \frac{\mathbf{k}}{r^3}(z-z')$$

$$= \frac{r}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right)$$

where S is the surface enclosing volume V', within which current $\mathbf{J}$ is confined.

We write then

$$\vec{\nabla}.\mathbf{A} = -\frac{\mu_0}{4\pi}\left(\int_S \frac{\mathbf{J}}{r}.d\mathbf{S} - \int_{V'} \frac{1}{r}(\vec{\nabla}'.\mathbf{J})\,dV'\right)$$

Everywhere on the surface S, which encloses the volume V', $\mathbf{J}$ is either zero or tangential so that

$$\mathbf{J}.d\mathbf{S} = 0,$$

in both the cases. Therefore

$$\vec{\nabla}.\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{1}{r}(\vec{\nabla}'.\mathbf{J})\,dV'. \qquad \text{...(3)}$$

From the equation of continuity,

$$\vec{\nabla}'.\mathbf{J} = -\frac{\partial\rho}{\partial t},$$

so that equation (3) becomes

$$\vec{\nabla}.\mathbf{A} = -\frac{\mu_0}{4\pi}\int_{V'} \frac{1}{r}\frac{\partial\rho}{\partial\tau}\,dV'$$

$$= -\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{V'} \frac{\rho}{r}\,dV', \qquad \text{...(4)}$$

where V' is any volume within which the current $\mathbf{J}$ is confined. Refer to chapter 1, from where we can write the potential ϕ, produced by a continuous charge distribution, as

$$\phi = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho\,dV'}{r}.$$

Using above relation, equation (4) can be written as

$$\vec{\nabla}.\mathbf{A} = -\mu_0\varepsilon_0\frac{\partial\phi}{\partial t} \qquad \text{...(5)}$$

which is called Lorentz condition for the vector potential $\mathbf{A}$ associated with the magnetic field $\mathbf{B}$ due to non-steady currents.

If ϕ is constant, then from eq. (5), we have

$$\vec{\nabla}.\mathbf{A} = 0. \qquad \text{...(6)}$$

This is the Lorentz condition for vector potential associated with steady currents.